

Algorithmic Complexity of Proper Labeling Problems

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Abstract

A proper labeling of a graph is an assignment of integers to some elements of a graph, which may be the vertices, the edges, or both of them, such that we obtain a proper vertex coloring via the labeling subject to some conditions. The problem of proper labeling offers many variants and received a great interest during recent years. We consider the algorithmic complexity of some variants of the proper labeling problems, we present some polynomial time algorithms and **NP**-completeness results for them.

Key words: Proper labeling; Computational complexity; Multiplicative vertex-coloring weightings; Gap vertex-distinguishing edge colorings ; Fictional coloring; Vertex-labeling by maximum; 1, 2, 3-Conjecture; Multiplicative 1, 2, 3-Conjecture.

Subject classification: 05C15, 05C20, 68Q25

1 Introduction

A proper labeling of a graph is an assignment of integers to some elements of a graph, which may be the vertices, the edges, or both of them, such that we obtain a proper vertex coloring via the labeling subject to some conditions. Karoński, Łuczak and Thomason initiated the study of proper labeling [19]. They introduced an edge-labeling which is additive vertex-coloring that means for every edge uv , the sum of labels of the edges incident to u is different from the sum of labels of the edges incident to v [19]. The problem of proper labeling offers many variants and received a great interest during recent years, for instance see [1, 9, 10, 18, 19, 24].

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There is a concept which has a close relationship with proper labeling problem. An *adjacent vertex-distinguishing edge coloring*, or *avd-coloring*, of a graph G is a proper edge coloring of G such that no pair of adjacent vertices meets the same set of colors. Avd-coloring has been widely studied, for example see [6, 8, 16].

In this paper, we consider the algorithmic complexity of the following proper labeling problems. We will present some polynomial time algorithms and **NP**-completeness results for them. Throughout the paper we denote $\{1, 2, \dots, k\}$ by \mathbb{N}_k .

Edge-labeling by sum.

An edge-labeling f is an *edge-labeling by sum* if $c(v) = \sum_{e \ni v} f(e)$ is a proper vertex coloring. This parameter was introduced by Karoński et al. and it is conjectured that three labels $\{1, 2, 3\}$ are sufficient for every connected graph, except K_2 (1, 2, 3-Conjecture, see [19]). This labeling has been studied extensively by several authors, for instance see [1, 2, 7, 20, 21, 24]. Currently, we know that every connected graph has an *edge-labeling by sum*, using the labels from \mathbb{N}_5 [18]. Furthermore, Addario-Berry, Dalal, and Reed showed that almost all graphs have an *edge-labeling by sum* from \mathbb{N}_2 [2].

Dudek and Wajc showed that determining whether a given graph has an *edge-labeling by sum* from \mathbb{N}_2 is **NP**-complete [14]. They showed that determining whether a graph has an *edge-labeling by sum* from either $\{0, 1\}$ or $\{1, 2\}$ is **NP**-complete. They conjectured that deciding whether a given graph has an *edge-labeling by sum* from $\{a, b\}$ for any rational a and b is **NP**-complete. By a short proof we improve the current results and show that for a given 3-regular graph G , determining whether G has an *edge-labeling by sum* from $\{a, b\}$ is **NP**-complete.

Theorem 1 *For a given 3-regular graph G , determining whether G has an edge-labeling by sum from \mathbb{N}_2 is **NP**-complete.*

Vertex-labeling by sum (Lucky labeling and sigma coloring).

A vertex-labeling f is a *vertex-labeling by sum* if $c(v) = \sum_{u \sim v} f(u)$ is a proper vertex coloring. *Vertex-labeling by sum* is a vertex version of the above problem, which was introduced recently by Czerwiński et al. [10]. It was conjectured that every graph G has a *vertex-labeling by sum*, using the labels $\{1, 2, \dots, \chi(G)\}$ [10] and it was shown that every graph G with $\Delta(G) \geq 2$, has a *vertex-labeling by sum*, using the labels $\{1, 2, \dots, \Delta^2 - \Delta + 1\}$ [4]. A similar version of this labeling was introduced by Chartrand et al. [9].

It was shown that, it is **NP**-complete to decide for a given planar 3-colorable graph G , whether G has a *vertex-labeling by sum* from \mathbb{N}_2 [3]. Furthermore, it is **NP**-complete to determine for a 3-regular graph G , whether G has a *vertex-labeling by sum* from \mathbb{N}_2 [12].

Edge-labeling by product. (Multiplicative vertex-coloring)

An edge-labeling f is an *edge-labeling by product* if $c(v) = \prod_{e \ni v} f(e)$ is a proper vertex coloring. This variant was introduced by Skowronek-Kaziów and it is conjectured that every non-trivial graph G has an *edge-labeling by product*, using the labels from \mathbb{N}_3 (Multiplicative 1, 2, 3-Conjecture, see [25]). Currently, we know that every non-trivial graph has an *edge-labeling by product*, using the labels from \mathbb{N}_4 [25]. Also, every non-trivial, 3-colorable graph G permits an *edge-labeling by product* from \mathbb{N}_3 [25]. We will prove the following for planar 3-colorable graphs and 3-regular graphs.

Theorem 2 *We have the following:*

- (i) *For a given planar 3-colorable graph G , determining whether G has an edge-labeling by product from \mathbb{N}_2 is **NP**-complete.*
- (ii) *For a given 3-regular graph G , determining whether G has an edge-labeling by product from \mathbb{N}_2 is **NP**-complete.*

Vertex-labeling by product.

A vertex-labeling f is a *vertex-labeling by product* if $c(v) = \prod_{u \sim v} f(u)$ is a proper vertex coloring. *Vertex-labeling by product* is a vertex version of the above problem.

For a given graph G , let $g : V(G) \rightarrow \{1, \dots, \chi(G)\}$ be a proper vertex coloring of G . Label the set of vertices $g^{-1}(1)$ by 1; also, for each i , $1 < i \leq \chi(G)$ label the set of vertices $g^{-1}(i)$ by the $(i-1)$ -th prime number; this labeling is a *vertex-labeling by product*. In number theory, Prime Number Theorem describes the asymptotic distribution of the prime numbers. As a consequence of Prime Number Theorem we have the following bound for the size of the n -th prime number p_n (i.e., $p_1 = 2$, $p_2 = 3$, etc.): $p_n < n \ln n + n \ln \ln n$, for $n \geq 6$ (see [5] p. 233). So, every graph G has a *vertex-labeling by product*, from $\{1, 2, \dots, \chi \ln \chi + \chi \ln \ln \chi + 2\}$. Here, we ask the following question.

Problem 1. *Does every graph G have a vertex-labeling by product, using the labels $\{1, 2, \dots, \chi(G)\}$?*

Every bipartite graph and 3-colorable graph has a *vertex-labeling by product* from \mathbb{N}_2 and \mathbb{N}_3 , respectively. We will prove the following about the planar 3-colorable graphs.

Theorem 3 *We have the following:*

- (i) *For a given planar 3-colorable graph G , determining whether G has a vertex-labeling by product from \mathbb{N}_2 is **NP**-complete.*
- (ii) *For every k , $k \geq 3$, it is **NP**-complete to determine whether a given graph has a vertex-labeling by product from \mathbb{N}_k .*

Also, we prove the following about random graphs.

Theorem 4 *For every constant p , $p \in (0, 1)$, almost all graphs in $G(n, p)$ have a vertex-labeling by product from \mathbb{N}_{11} .*

Let G be a 3-regular graph. G has a *vertex-labeling by product* from \mathbb{N}_2 if and only if G has a *vertex-labeling by sum* from \mathbb{N}_2 . It was proved that it is **NP**-complete to determine for a given 3-regular graph G , whether G has a *vertex-labeling by sum* from \mathbb{N}_2 [12]. So we have the following:

Theorem A *It is NP-complete to determine for a given 3-regular graph G , whether G has a vertex-labeling by product from \mathbb{N}_2 .*

Edge-labeling by gap.

An edge-labeling f is an *edge-labeling by gap* if

$$c(v) = \begin{cases} 1 & \text{if } d(v) = 0, \\ f(e)_{e \ni v} & \text{if } d(v) = 1, \\ \max_{e \ni v} f(e) - \min_{e \ni v} f(e) & \text{otherwise,} \end{cases}$$

is a proper vertex coloring. Every graph G has an *edge-labeling by gap* if and only if it has no connected component isomorphic to K_2 ; for instance, put the different powers of two on the edges of G .

A similar definition was introduced by Tahraoui et al. [26]. They introduced the following variant: Let G be a graph, k be a positive integer and f be a mapping from $E(G)$ to the set \mathbb{N}_k . For each vertex v of G , the label of v is defined as

$$c(v) = \begin{cases} f(e)_{e \ni v} & \text{if } d(v) = 1, \\ \max_{e \ni v} f(e) - \min_{e \ni v} f(e) & \text{otherwise,} \end{cases}$$

The mapping f is called *gap vertex-distinguishing labeling* if distinct vertices have distinct labels. Such a coloring is called a *gap- k -coloring* and is denoted by $\text{gap}(G)$ [26]. It was conjectured that for a connected graph G of order n with $n > 2$, $\text{gap}(G) \in \{n - 1, n, n + 1\}$ [26]. They propose study of the variant of the gap coloring problem that distinguishes the adjacent vertices only.

Now, consider the following example.

Remark 1 Every complete graph K_n of order n with $n > 2$, has an *edge-labeling f_n by gap* from $\{1, 2, \dots, \chi(K_n) + 1\}$. Suppose that $K_3 = v_1v_2v_3$ and let f_3 be the following function: $f_3(v_1v_2) = 4$, $f_3(v_1v_3) = 1$ and $f_3(v_2v_3) = 2$. Define f_n recursively:

$$f_n(v_i v_j) = \begin{cases} f_{n-1}(v_i v_j) + 1 & \text{if } i, j < n, \\ 1 & \text{if } (i = n \text{ and } j \neq 2) \text{ or } (j = n \text{ and } i \neq 2), \\ 2 & \text{otherwise.} \end{cases}$$

Let f be an *edge-labeling by gap* from \mathbb{N}_k , for a graph G we have $k \geq \chi(G) - 1$. Now, we state the following problem:

Problem 2. *Does every connected graph G of order n with $n > 2$, have an edge-labeling by gap from $\{1, 2, \dots, \chi(G) + 1\}$?*

We will prove the following:

Theorem 5 *We have the following:*

- (i) *For a given planar bipartite graph G with minimum degree two, determining whether G has an edge-labeling by gap from \mathbb{N}_2 is in \mathbf{P} .*
- (ii) *For a given planar bipartite graph G , determining whether G has an edge-labeling by gap from \mathbb{N}_2 is \mathbf{NP} -complete.*
- (iii) *For every k , $k \geq 3$, it is \mathbf{NP} -complete to determine whether a given graph has an edge-labeling by gap from \mathbb{N}_k .*

Vertex-labeling by gap.

A vertex-labeling f is a *vertex-labeling by gap* if

$$c(v) = \begin{cases} 1 & \text{if } d(v) = 0, \\ f(u)_{u \sim v} & \text{if } d(v) = 1, \\ \max_{u \sim v} f(u) - \min_{u \sim v} f(u) & \text{otherwise,} \end{cases}$$

is a proper vertex coloring. A graph may lack any *vertex-labeling by gap*. Here we ask the following:

Problem 3. *Does there exist a polynomial time algorithm to determine whether a given graph has a vertex-labeling by gap?*

Every tree T has a *vertex-labeling by gap* from \mathbb{N}_2 . Let T be a tree and $v \in V(T)$ be an arbitrary vertex, define:

$$f(u) = \begin{cases} 1 & \text{if } d(u, v) \equiv 0 \pmod{4}, \\ 2 & \text{otherwise.} \end{cases}$$

It is easy to see that this labeling is a *vertex-labeling by gap* from \mathbb{N}_2 .

We will show that there is dichotomy for the problem determining whether a given graph has a *vertex-labeling by gap* from \mathbb{N}_2 : it is polynomially solvable for planar bipartite graphs but **NP**-complete for bipartite graphs.

Theorem 6 *We have the following:*

- (i) *For a given planar bipartite graph G , determining whether G has a vertex-labeling by gap from \mathbb{N}_2 is in **P**.*
- (ii) *For a given bipartite graph G , determining whether G has a vertex-labeling by gap from \mathbb{N}_2 is **NP**-complete.*
- (iii) *For every k , $k \geq 3$, it is **NP**-complete to determine whether a given graph has a vertex-labeling by gap from \mathbb{N}_k .*
- (iv) *It is **NP**-complete to decide whether a given planar 3-colorable graph G has a vertex-labeling by gap from \mathbb{N}_2 .*

Note that, every bipartite graph $G = [X, Y]$ has a *vertex-labeling by gap*, label the set of vertices X by 1 and label the set of vertices Y by different even numbers.

Remark 2 A hypergraph H is a pair (X, Y) , where X is the set of vertices and Y is a set of non-empty subsets of X , called edges. The k -coloring of H is a coloring $\ell : X \rightarrow \mathbb{N}_k$ such that, for every edge e with $|e| > 1$, there exist $v, u \in X$ such that $\ell(u) \neq \ell(v)$. It was shown by Thomassen [27] that, for any k -uniform and k -regular hypergraph H , if $k \geq 4$, then H is 2-colorable. For a given r -regular bipartite graph $G = [X, Y]$ with $r > 3$, consider the hypergraph H with the vertex set X and edge set Y such that $v \in e$ in H if and only if $v \in X$ is adjacent to $e \in Y$ in G . Let ℓ be a 2-coloring for H . Define

$$c(v) = \begin{cases} \ell(v) & \text{if } v \in X, \\ 2 & \text{otherwise,} \end{cases}$$

this labeling is *vertex-labeling by gap* from \mathbb{N}_2 . So every r -regular bipartite graph $G = [X, Y]$ with $r \geq 4$, has a *vertex-labeling by gap* from \mathbb{N}_2 . But, there are 3-uniform, 3-regular hypergraphs that are not 2-colorable. For instance, consider the Fano Plane. The Fano Plane is a hypergraph with seven vertices \mathbb{Z}_7 and seven edges $\{\{i, i+1, i+3\} : 1 \leq i \leq 7\}$.

Problem 4. *Determine the computational complexity of deciding whether a given 3-regular bipartite graph G have a vertex-labeling by gap from \mathbb{N}_2 .*

Vertex-labeling by degree. (Fictional coloring)

A vertex-labeling f is a *vertex-labeling by degree* if $c(v) = f(v)d(v)$, where $d(v)$ is the

degree of vertex v , is a proper vertex coloring. This parameter was introduced by Bosek, Grytczuk, Matecki and Żelazny [30]. They conjecture that every graph G has a *vertex-labeling by degree* from $\{1, 2, \dots, \chi(G)\}$. Let p be a prime number and let G be a graph such that $\chi(G) \leq p - 1$, they proved that G has a *vertex-labeling by degree* from \mathbb{N}_{p-1} , so every graph G has a *vertex-labeling by degree* from $\{1, 2, \dots, 2\chi(G)\}$ [30].

Since determining the chromatic number of regular graph is **NP**-hard, hence for a given regular graph G , determining the minimum k such that G has a *vertex-labeling by degree* from \mathbb{N}_k is **NP**-hard. We will prove the following:

Theorem 7

- (i) *Determining whether a given graph has a vertex-labeling by degree from \mathbb{N}_2 is in **P**.*
- (ii) *For every $k, k \geq 3$, for a given graph G with $\chi(G) = k + 1$ determining whether G has vertex-labeling by degree from \mathbb{N}_k is **NP**-complete.*

Vertex-labeling by maximum.

A vertex-labeling f is a *vertex-labeling by maximum* if $c(v) = \max_{u \sim v} f(u)$ is a proper vertex coloring. A graph G may lack any *vertex-labeling by maximum* and it has a *vertex-labeling by maximum* from \mathbb{N}_2 if and only if G is bipartite.

Remark 3 Let k be the minimum number such that G has a *vertex-labeling by maximum* from the set \mathbb{N}_k , then $k - \chi(G)$ can be arbitrarily large. For instance, for a given $t, t > 3$ consider the graph G with vertex set $V(G) = \{a_i : 1 \leq i \leq t\} \cup \{b_j : 1 \leq j \leq t - 2\}$ and edge set $E(G) = \{a_i a_{i+1} : 1 \leq i \leq t - 1\} \cup \{a_j b_j, b_j a_{j+1} : 1 \leq j \leq t - 2\}$. G is 3-colorable but $k = t$.

Every triangle-free graph has a *vertex-labeling by maximum* (put different numbers on vertices) and if G is a graph such that every vertex appears in some triangles, then G does not have any *vertex-labeling by maximum*. Here, we present a nontrivial necessary condition that can be checked in polynomial time for a graph to have a *vertex-labeling by maximum*. First consider the following definition.

Definition 1 For a given graph G the subset S of vertices is called *triangular-structured-vertices (TSV)* if every $v \in S$ appears in a triangle in $G[S]$ and for every two adjacent vertices v and u , where $v \in S$ and $u \notin S$, there exists a vertex $z \in S$ such that z is adjacent to v and u .

Let S be a TSV for G . By way of contradiction, assume that f is a *vertex-labeling by maximum* for G and $v \in S \cup N(S)$ is a vertex such that $f(v) = \max_{u \in S \cup N(S)} f(u)$.

Then v has two neighbors x and y in S with $\max_{u \sim x} f(u) = \max_{u \sim y} f(u) = f(v)$. This is a contradiction. Therefore, if G has a TSV, then G does not have a *vertex-labeling by maximum*.

Theorem 8 *We have the following:*

- (i) *For a given 3-regular graph G , determining whether G has a vertex-labeling by maximum from \mathbb{N}_3 is **NP**-complete.*
- (ii) *For every k , $k \geq 3$, it is **NP**-complete to decide whether G has a vertex-labeling by maximum from \mathbb{N}_k for a given k -colorable graph G .*
- (iii) *There is a polynomial time algorithm to determine whether G has a TSV.*

Here, we ask the following questions:

Problem 5. *Is the necessary condition, sufficient for a given graph to have a vertex-labeling by maximum?*

Problem 6. *Does there exist a polynomial time algorithm to determine whether a given graph has a vertex-labeling by maximum?*

Notation

Throughout this paper all graphs are finite and simple. We follow [15, 29] for terminology and notations are not defined here, and we consider finite undirected simple graphs $G = (V(G), E(G))$. We denote a 3-Sat formula by $\Phi = (X, C)$, where X is the set of variables and C is a set of clauses over X such that each clause $c \in C$ has $|c| = 3$.

A *proper vertex coloring* of G is a function $c : V(G) \rightarrow L$, such that if $u, v \in V(G)$ are adjacent, then $c(u)$ and $c(v)$ are different. A *proper vertex k -coloring* is a proper vertex coloring with $|L| = k$. The smallest integer k such that G has a proper vertex k -coloring is called the *chromatic number* of G and denoted by $\chi(G)$. Similarly, for $k \in \mathbb{N}$, a *proper edge k -coloring* of G is a function $c : E(G) \rightarrow \mathbb{N}_k$, such that if $e, e' \in E(G)$ share a common endpoint, then $c(e)$ and $c(e')$ are different. The smallest integer k such that G has a proper edge k -coloring is called the *edge chromatic number* of G and is denoted by $\chi'(G)$. By Vizing's theorem [28], the edge chromatic number of a graph G is equal to either $\Delta(G)$ or $\Delta(G) + 1$. Those graphs G for which $\chi'(G) = \Delta(G)$ are said to belong to *Class 1*, and the others to *Class 2*.

We denote the induced subgraph G on S by $G[S]$. Also, for every $v \in V(G)$ and $S \subseteq V(G)$, $N(v)$ and $N(S)$ denote the neighbor set of v and the set of vertices of G which

has a neighbor in S , respectively. Furthermore, let H be a subgraph of G , for every vertex v , $v \in V(H)$, we denote $|\{u : vu \in E(H), u \in V(H)\}|$ by $d_H(v)$.

Summery of results

Table 1: Recent results on proper labeling of graphs

Edge	NP -h	P	Upper Bound	Conjecture
Sum	3-regular	-	\mathbb{N}_5	\mathbb{N}_3
Product	3-regular Planar 3-col	-	\mathbb{N}_4	\mathbb{N}_3
Gap	Planar bipartite	Planar bipartite with $\delta > 1$	-	$\mathbb{N}_{\chi+1}$
Vertex				
Sum	3-regular Planar 3-col	-	$\mathbb{N}_{\Delta^2-\Delta+1}$	\mathbb{N}_χ
Product	3-regular Planar 3-col	-	$\mathbb{N}_{\chi \ln \chi + \chi \ln \ln \chi + 2}$	\mathbb{N}_χ
Degree	4-col	From \mathbb{N}_2	$\mathbb{N}_{2\chi}$	\mathbb{N}_χ
Maximum	3-regular	Bipartite	-	-
Gap	Bipartite Planar 3-col	Planar Bipartite	-	-

2 Proof of Theorem 1

We reduce *Monotone Not-All-Equal (NAE) 3-Sat* to our problem in polynomial time. It is shown that the following problem is **NP**-complete [15].

Monotone Not-All-Equal 3-Sat .

INSTANCE: A 3-Sat formula (X, C) such that there is no negation in the formula.

QUESTION: Is there a truth assignment for X such that each clause in C has at least one true literal and at least one false literal?

Without loss of generality suppose that Φ is a formula such that every variable x , $x \in X$



Figure 1: The gadgets B_x . In every *edge-labeling* f by *sum* from \mathbb{N}_2 , $f(e) = f(e')$.

appears in more than one clause. For every variable x , $x \in X$, we denote the number of clauses which contains x by $\gamma(x)$. For a given formula Φ we construct a 3-regular graph G , such that Φ has a NAE truth assignment if and only if G has an *edge-labeling by sum* from \mathbb{N}_2 . For every variable x if $\gamma(x) > 2$, let A_x be a cycle of length $\gamma(x)$. Otherwise let A_x be two vertices a and b with two parallel edges between a and b . Construct the graph B_x by replacing every edge ab of A_x by $I(a, b)$ which is shown in Figure 1. Let $c_{i_1}, \dots, c_{i_{\gamma(x)}}$ be the clauses which contains x . B_x has $\gamma(x)$ vertices of degree two, call these vertices $x_{i_1}, \dots, x_{i_{\gamma(x)}}$, in an arbitrary order. Now, for every variable x , $x \in X$, put a copy of B_x and for every clause c_i , $c_i \in C$, put a vertex c_i . For every clause c_i and every variable x , if x appears in c_i , then join c_i to x_i . Clearly, G is a simple 3-regular graph (see Figure 2 for more details).

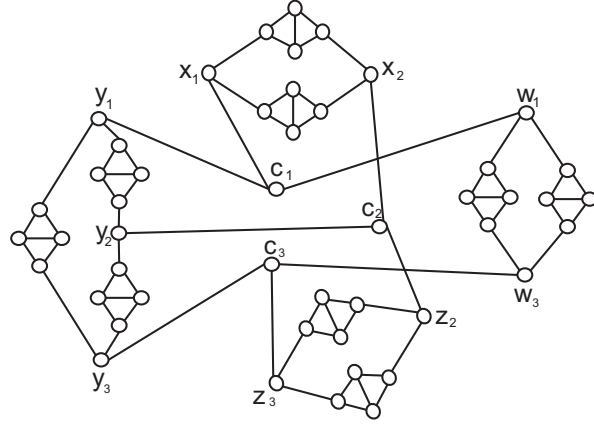


Figure 2: The graph G derived from the formula $\Phi = c_1 \wedge c_2 \wedge c_3$, where $c_1 = x \vee y \vee w$, $c_2 = x \vee y \vee z$ and $c_3 = y \vee w \vee z$.

First assume that G has an *edge-labeling* f by *sum* from \mathbb{N}_2 and let ℓ be the coloring which is induced by f . By the structures of A_x and B_x , for a variable x which appears in $c_{i_1}, \dots, c_{i_{\gamma(x)}}$ the labels of edges $c_{i_1}x_{i_1}, \dots, c_{i_{\gamma(x)}}x_{i_{\gamma(x)}}$ are the same. Now, for every variable x , which appears in $c_{i_1}, \dots, c_{i_{\gamma(x)}}$ put $\Gamma(x) = \text{true}$ if and only if the label of

edge $c_{i_1}x_{i_1}$ are 2. Let c_i be a clause with the variable x, y, z . First, suppose that $\{f(x_i c_i), f(y_i c_i), f(z_i c_i)\} \neq \{1, 2\}$, then $\ell(x) = \ell(c_i)$, but this is a contradiction. So, we have $\{f(x_i c_i), f(y_i c_i), f(z_i c_i)\} = \{1, 2\}$. Consequently, Γ is a NAE satisfying assignment. Next, suppose that Φ has a NAE satisfying assignment $\Gamma : X \rightarrow \{\text{true}, \text{false}\}$, for every variable x , which appears in $c_{i_1}, \dots, c_{i_{\gamma(x)}}$ label all the edges $c_{i_1}x_{i_1}, \dots, c_{i_{\gamma(x)}}x_{i_{\gamma(x)}}$ by 2 if and only if $\Gamma(x) = \text{true}$. It is easy to extend this labeling to an *edge-labeling by sum* from \mathbb{N}_2 . The proof is complete. ♣

3 Proof of Theorem 2

(i) Clearly, the problem is in **NP**. We reduce *Cubic Planar 1-In-3 3-Sat* to our problem. Moore and Robson [22] proved that the following problem is **NP**-complete.

Cubic Planar 1-In-3 3-Sat.

INSTANCE: A 3-Sat formula $\Phi = (X, C)$ such that every variable appears in exactly three clauses, there is no negation in the formula, and the bipartite graph obtained by linking a variable and a clause if and only if the variable appears in the clause, is planar.

QUESTION: Is there a truth assignment for X such that each clause in C has exactly one true literal?

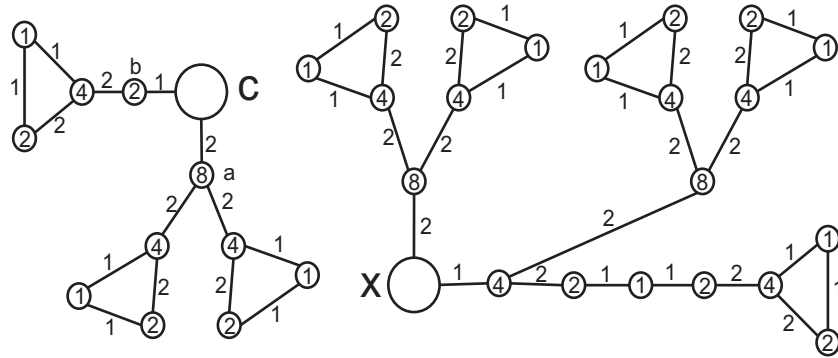


Figure 3: The two gadgets H_x and I_c . I_c is on the left hand side of the figure.

Consider an instance Φ of *Cubic Planar 1-In-3 3-Sat*. We transform this into a planar 3-colorable graph G_Φ such that G_Φ has an *edge-labeling by product* from \mathbb{N}_2 if and only if Φ has a 1-in-3 assignment. We use two gadgets H_x and I_c which are shown in Figure 3. The graph G_Φ has a copy of H_x for each variable $x \in X$ and a copy of I_c for each clause $c \in C$. Also, for each clause $c = y \vee z \vee w$ add the edges cy , cz and cw . First, suppose that G_Φ has an *edge-labeling by product* from \mathbb{N}_2 . In every copy of H_x and I_c the label of

every edge is determined. In Figure 3, the label of each edge is written on the edge and the color of each vertex induced by edge labels is written on the vertex. Every variable x appears in exactly three clauses, suppose that x appears in c_i, c_j and c_k . By attention to the induced colors of neighbors of x in H_x , the labels of edges $c_i x, c_j x$ and $c_k x$ are equal. Furthermore, by attention to the H_x and I_c , for every clause $c = x \vee y \vee z$, the label of exactly one of the edges cx, cy and cz is 2 (note that if the labels of edges cx, cy and cz are 2, then the induced colors of x and c are same). Now, for every variable x , which appears in c_i, c_j and c_k put $\Gamma(x) = \text{true}$ if and only if the labels of edges $c_i x, c_j x$ and $c_k x$ are 2. Clearly, Γ is a 1-in-3 satisfying assignment. Next, suppose that Φ has a 1-in-3 satisfying assignment $\Gamma : X \rightarrow \{\text{true}, \text{false}\}$, for every variable x , which appears in c_i, c_j and c_k , label $c_i x, c_j x$ and $c_k x$ by 2 if and only if $\Gamma(x) = \text{true}$. It is easy to extent this labeling to an *edge-labeling by product* from \mathbb{N}_2 .

(ii) Let G be a 3-regular graph. G has an *edge-labeling by product* from \mathbb{N}_2 if and only if G has an *edge-labeling by sum* from \mathbb{N}_2 , so by Theorem 1, we can prove the theorem. ♣

4 Proof of Theorem 3

(i) Clearly, the problem is in **NP**. We reduced *Cubic Planar 1-In-3 3-Sat* to our problem. In [22] it was proved that *Cubic Planar 1-In-3 3-Sat* is **NP**-complete. First, we construct an auxiliary graph \mathcal{H}_i^c . Put a copy of triangle $K_3 = z_1^c z_2^c z_3^c$. For every vertex $z_j^c, 1 \leq j \leq 2$, put $2i$ new isolated vertices $t_1^j, t_2^j, \dots, t_{2i}^j$ and join z_j^c to all of them. Also, add the edges $t_1^j t_2^j, t_3^j t_4^j, \dots, t_{2i-1}^j t_{2i}^j$. Next, put $2i - 2$ new isolated vertices $t_1^3, t_2^3, \dots, t_{2i-2}^3$ and join z_3^c to all of them. Finally, add the edges $t_1^3 t_2^3, t_3^3 t_4^3, \dots, t_{2i-3}^3 t_{2i-2}^3$. Call the resulting graph \mathcal{H}_i^c . Now, consider an instance Ψ , we transform this into a planar 3-colorable graph G_Ψ such that G_Ψ has a *vertex-labeling by product* from \mathbb{N}_2 if and only if Ψ has a 1-in-3 assignment. For each clause $c \in C$ put a vertex c and a copy of $\mathcal{H}_3^c, \mathcal{H}_5^c$ and \mathcal{H}_6^c . Also, join the vertex c to $z_3^c \in \mathcal{H}_3^c, z_5^c \in \mathcal{H}_5^c$ and $z_6^c \in \mathcal{H}_6^c$. Next, for each variable $x \in X$ put a vertex x . Finally, for each clause $c = x \vee y \vee w$ add the edges cx, cy and cw .

First, suppose that G_Ψ has a *vertex-labeling by product* from \mathbb{N}_2 and let ℓ be the induced coloring by f . In every copy of \mathcal{H}_3^c the label of vertex z_3^c is 2. We have the similar property for \mathcal{H}_5^c and \mathcal{H}_6^c . By the structure of \mathcal{H}_3^c , we have $f(c) = 1$ and in the subgraph \mathcal{H}_3^c , $\ell(z_3^c) = 8$; similarly for every subgraph \mathcal{H}_5^c , $\ell(z_5^c) = 32$ and for each subgraph \mathcal{H}_6^c , we have $\ell(z_6^c) = 64$. So for every clause vertex c we have $\ell(c) = 16$. Now, for every variable x , put $\Gamma(x) = \text{true}$ if and only if $f(x) = 2$. Since for every clause c , $\ell(c) = 16$, Γ is a 1-in-3 satisfying assignment. On the other hand, suppose that Ψ is 1-in-3 satisfiable with the satisfying assignment $\Gamma : X \rightarrow \{\text{true}, \text{false}\}$, for every variable x , label the vertex x by 2

if and only if $\Gamma(x) = \text{true}$. The labels of other vertices are determined and it is clear that this labeling is a *vertex-labeling by product* from \mathbb{N}_2 .

(ii) For every k , $k \geq 3$, we present a polynomial time reduction from *3-colorability* to our problem.

3-Colorability

INSTANCE: A graph G .

QUESTION: Is $\chi(G) \leq 3$?

First define the following sets: $\mathcal{A}_k = \{mn : m, n \in \mathbb{N}_k\}$, $\mathcal{B}_k = \{\frac{m}{n} : m, n \in \mathbb{N}_k\}$. Also, define $\alpha(k) = \max_{\mathcal{D}_k \in \mathcal{C}_k} |\mathcal{D}_k|$, where \mathcal{C}_k is the set of sets such that for every set $\mathcal{D}_k \in \mathcal{C}_k$, we have $\mathcal{D}_k \subseteq \mathcal{A}_k$ and $\{\frac{d}{d'} : d, d' \in \mathcal{D}_k\} \cap \mathcal{B}_k = \emptyset$. Since k is constant, so we can compute $\alpha(k)$ in $O(1)$. Note that for k , $k \geq 3$, $\alpha(k) \geq 3$. Now, for a given graph G with n vertices v_1, v_2, \dots, v_n , join all vertices of G to the all vertices of complete graph $K_{\alpha(k)-3}$ with the vertices $v_{n+1}, \dots, v_{n+\alpha(k)-3}$. Call the resulting graph G^* . Now consider the graph G^{**} with the vertex set $\{v_i^j : i \in \mathbb{N}_{n+\alpha(k)-3}, j \in \mathbb{N}_k\}$ such that v_x^y is joined to v_z^w if and only if $x = z$ or $v_x v_z \in E(G^*)$. Finally, consider a copy of graph G^{**} , for every i , $1 \leq i \leq n + \alpha(k) - 3$, put two new isolated vertices v_i' and v_i'' and join them to the set of vertices $\{v_i^1, \dots, v_i^k\}$. Call the resulting graph \tilde{G} .

We show that \tilde{G} has a *vertex-labeling by product* from \mathbb{N}_k if and only if G is 3-colorable. Let f be a *vertex-labeling by product* for \tilde{G} . Clearly, $f(v_1^1), \dots, f(v_1^k)$ should be different integers. For every i , $i \in \mathbb{N}_{n+\alpha(k)-3}$, we have: $\{f(v_i^j) : j \in \mathbb{N}_k\} = \mathbb{N}_k$. Furthermore, for every i_1, i_2 , $1 \leq i_1 < i_2 \leq n + \alpha(k) - 3$, we have: $f(v_{i_1}')f(v_{i_1}''), f(v_{i_2}')f(v_{i_2}'') \in \mathcal{A}_k$. Also, for every i_1 and i_2 , if $v_{i_1} v_{i_2} \in E(G^*)$, then

$$\frac{f(v_{i_1}')f(v_{i_1}'')}{f(v_{i_2}')f(v_{i_2}'')} \notin \mathcal{B}_k.$$

Therefore, $|\{f(v_i')f(v_i'') : 1 \leq i \leq n + \alpha(k) - 3\}| \geq \alpha(k) - 3 + \chi(G)$. So, \tilde{G} has a *vertex-labeling by product* from \mathbb{N}_k if and only if $\chi(G) \leq 3$. The proof is complete. ♣

5 Proof of Theorem 4

For a given $G(n, p)$, let $\{\mathcal{A}_1, \dots, \mathcal{A}_5\}$ be a partition of vertices to five parts such that those parts are of equal or almost equal sizes (that is, $\lfloor n/5 \rfloor$ or $\lceil n/5 \rceil$). For every vertex v denote the number of neighbors of v in \mathcal{A}_i by λ_v^i . Label the vertices of $\mathcal{A}_1, \dots, \mathcal{A}_5$ by 2, 3, 5, 7, 11, respectively. This labeling is a *vertex-labeling by product* if for every two adjacent vertices v and u , there is an index i , $1 \leq i \leq 5$ such that $\lambda_v^i \neq \lambda_u^i$.

$$\begin{aligned}
Pr(\lambda_v^i = \lambda_u^i) &\leq \Theta\left(\sum_{t=0}^{n/5} \left(\binom{n/5}{t} p^t (1-p)^{n/5-t}\right)^2\right) \\
&\leq \Theta\left(\max_{0 \leq t \leq n/5} \binom{n/5}{t} p^t (1-p)^{n/5-t}\right) \\
&= \Theta\left(\binom{n/5}{pn/5} p^{pn/5} (1-p)^{(1-p)n/5}\right).
\end{aligned}$$

By Stirling's approximation $\sqrt{2\pi n}(\frac{n}{e})^n \leq n! \leq \sqrt{e^2 n}(\frac{n}{e})^n$, we have:

$$\begin{aligned}
Pr(\lambda_v^i = \lambda_u^i) &= \Theta(n^{-\frac{1}{2}}), \\
Pr(\forall i \lambda_v^i = \lambda_u^i) &= \Theta(n^{-\frac{5}{2}}), \\
Pr(\exists vu \forall i \lambda_v^i = \lambda_u^i) &= \Theta(n^2)\Theta(n^{-\frac{5}{2}}) = o(1).
\end{aligned}$$

The proof is complete. ♣

6 Proof of Theorem 5

(i) Moret proved in [23] that *Planar Not-All-Equal 3-Sat* is in **P** by an interesting reduction to a known problem in **P**, namely Planar(Simple) MaxCut.

Planar Not-All-Equal 3-Sat.

INSTANCE: A 3-Sat formula (X, C) such that the following graph obtained from 3-Sat is planar. The graph has one vertex for each variable, one vertex for each clause; all variable vertices are connected in a simple cycle and each clause vertex is connected by an edge to variable vertices corresponding to the literals present in the clause (note that positive and negative literals are treated exactly alike).

QUESTION: Is there a Not-All-Equal (NAE) truth assignment for X ?

By a simple argument it was shown that the following problem is in **P** (for more information see [12]).

Planar NAE 3-Sat Type 2.

INSTANCE: A 3-Sat formula (X, C) such that the following graph obtained from 3-Sat is planar. The graph has one vertex for each variable, one vertex for each clause and each

clause vertex is connected by an edge to variable vertices corresponding to the literals present in the clause.

QUESTION: Is there a NAE truth assignment for X ?

Now, consider the following variant.

Planar NAE Sat Type 2.

INSTANCE: Set X of variables, collection C of clauses over X such that each clause $c \in C$ has $|c| \geq 2$ and the following graph obtained from sat is planar. The graph has one vertex for each variable, one vertex for each clause and each clause vertex is connected by an edge to variable vertices corresponding to the literals present in the clause.

QUESTION: Is there a NAE truth assignment for X ?

We can convert any instance Φ of *Planar NAE Sat Type 2* to an instance Ψ of *Planar NAE 3-Sat Type 2* in polynomial time. For a given instance Φ , for each clause with exactly two literals like $c = (x \vee y)$, put two clauses $x \vee y \vee t$ and $x \vee y \vee \neg t$ in Ψ , where t is a new variable. Also, for each clause with exactly four literals $c = (x \vee y \vee w \vee z)$, put two clauses $x \vee y \vee t$ and $w \vee z \vee \neg t$ in Ψ , where t is a new variable. For clauses with more than five variables we have a similar argument.

For a given graph G , we can use a depth-first search algorithm to identify the connected components of G . G has an *edge-labeling by gap* from $\{1, 2\}$ if and only if each connected component of G has an *edge-labeling by gap* from $\{1, 2\}$. So, without loss of generality we can assume that G is connected. Let $G[X, Y]$ be a connected planar bipartite graph such that G does not have a vertex of degree one. Let f be an *edge-labeling by gap* from $\{1, 2\}$ for G . The color which is induced by f for the set of vertices X is one and the induced color by f for the set of vertices Y is zero, or vice versa (**Property 1**).

For every vertex $v \in X$, consider a variable v in Φ and for every vertex $u \in Y$ put a clause $(\bigvee_{v \sim u} v)$ in Φ . Now determine whether Φ has a NAE truth assignment. If Φ has a NAE truth assignment Γ , for every vertex v , $v \in X$ label all edge incident with v by 1 if and only if $\Gamma(v) = false$. Label other edges of G by 2, call this labeling f . It is easy to see that f is an *edge-labeling by gap*. If Φ does not have a NAE truth assignment. Then, for every vertex $v \in Y$, consider a variable v in Ψ and for every vertex $u \in X$ put a clause $(\bigvee_{v \sim u} v)$ in Ψ . Now determine whether Ψ has a NAE truth assignment. If Ψ has a NAE truth assignment Γ by a similar method we can find an *edge-labeling by gap* from $\{1, 2\}$ for G . If Ψ does not have a NAE truth assignment, by Property 1, G does not have any *edge-labeling by gap* from $\{1, 2\}$.

(ii) Let Φ be a 3-Sat formula with clauses $C = \{c_1, \dots, c_k\}$ and variables $X = \{x_1, \dots, x_n\}$. Let $G(\Phi)$ be a graph with the vertices $C \cup X \cup (\neg X)$, where $\neg X = \{\neg x_1, \dots, \neg x_n\}$, such

that for each clause $c_j = y \vee z \vee w$, c_j is adjacent to y, z and w , also every $x_i \in X$ is adjacent to $\neg x_i$. Φ is called planar 3-Sat type 2 formula if $G(\Phi)$ is a planar graph. It was shown that the problem of satisfiability of planar 3-Sat type 2 is **NP**-complete [13].

Planar 3-Sat type 2.

INSTANCE: A 3-Sat type 2 formula Φ .

QUESTION: Is there a truth assignment for Φ that satisfies all the clauses?

In *planar 3-Sat type 2*, if we only consider the set of formulas \mathcal{F} such that for every Φ , $\Phi \in \mathcal{F}$, $G(\Phi)$ is connected, the problem remains **NP**-complete. We reduce this version to our problem. Consider an instance Φ , we transform this into a graph G_Φ such that G_Φ has an *edge-labeling by gap* from \mathbb{N}_2 if and only if Φ has a satisfying assignment. For each variable $x \in X$ put a copy of cycle $C_4 = xt_x \neg xt'_x$. For each clause $c \in C$ put a copy of gadget $P_4 = cc'c''c'''$. Now, put a copy C_6 . Also, for each clause $c = y \vee z \vee w$ add the edges cy, cz and cw . Finally, let x be an arbitrary variable, join x to one of the vertices of C_6 . Call the resulting graph G_Φ . G_Φ is connected, bipartite and planar. First, suppose that G_Φ has an *edge-labeling f by gap* from \mathbb{N}_2 and l is the proper coloring which is induced by f . Since for every variable x the degrees of vertices x and $\neg x$ are greater than one, also for every clause c the degree of vertex c is 4 and G_Φ is connected, hence in the induced coloring l by f , for the set of variables and the set of clauses, we have $l(x_1) = l(\neg x_1) = \dots = l(\neg x_n) = l(x_n) \neq l(c_1) = \dots = l(c_m)$ and $l(x_1) \neq 2 \neq l(c_1)$.

First, suppose that $l(x) = 1$, since x is adjacent to one of the vertices of C_6 , in this situation G_Φ does not have any *edge-labeling f by gap* from \mathbb{N}_2 . So for all $x \in X$ and $c \in C$, we have $l(x) = 0$, $l(\neg x) = 0$ and $l(c) = 1$. Hence, the labels of all edges incident with x are same. Also, for every variable x , because of t_x , the labels of all edges incident with x are different from the labels of all edges incident with $\neg x$. Now, for every variable x , which appears in c_i, c_j, \dots, c_k put $\Gamma(x) = \text{true}$ if and only if the label of edge $c_i x$ is 2. For every clause $c = x \vee y \vee w$, $l(c) = 1$. If the labels of all edges cx, cy, cw are 1, then $f(cc') = 2$. Thus $f(c'c'') = 2$ and $f(c''c''') = 1$, so $l(c'') = l(c''')$, but this is a contradiction. Therefore, $2 \in \{f(cx), f(cy), f(cw)\}$. Therefore, Γ is an satisfying assignment. Now, let Γ be a satisfying assignment for Φ . For every variable x , label all the edges incident with x by 2 if and only if $\Gamma(x) = \text{true}$. It is easy to extend this labeling to an *edge-labeling f by gap* from $\{1, 2\}$. This completes the proof.

(iii) For every k , we present a polynomial time reduction from *k-colorability*, $k > 2$, to our problem.

k-Colorability

INSTANCE: A graph G .

QUESTION: Is $\chi(G) \leq k$?

For a given connected graph G with more than two vertices, we construct a graph G^* such that $\chi(G) \leq k$ if and only if G^* has an *edge-labeling by gap* from \mathbb{N}_k . For every vertex $v, v \in V(G)$, put a copy of gadget $P_3 = v'v''v'''$ and join v' to u' if and only if $vu \in E(G)$. Call the resulting graph G^* . First, suppose that G^* has an *edge-labeling by gap* from $\{1, 2, \dots, k\}$ and ℓ is the coloring which is induced by f . For every vertex v' , we have $\ell(v') \in \{0, 1, \dots, k-1\}$, so ℓ is a proper vertex coloring for G . Now, let c be a proper vertex coloring for G . For every vertex $v', v' \in V(G^*)$, label all edges incident with v' except $v'v''$ by 1 and label $v'v''$ by $c(v)$. Finally for every edge $v''v'''$, label $v''v'''$ by $c(v)$ if $c(v) \neq 1$, otherwise label $v''v'''$ by k . This labeling is an *edge-labeling by gap* from \mathbb{N}_k . ♣

7 Proof of Theorem 6

(i) For a given graph G , we can use a depth-first search algorithm to identify the connected components of G . G has a *vertex-labeling by gap* from $\{1, 2\}$ if and only if each connected component of G has a *vertex-labeling by gap* from $\{1, 2\}$. So, without loss of generality we can assume that G is connected. Let $G[X, Y]$ be a connected planar bipartite graph with more than two vertices. Consider a copy of G , remove all vertices of degree one from G and call the resulting graph $G'[X', Y']$, where $X' \subseteq X$. If $E(G') = \emptyset$, then G is a star, so G has a *vertex-labeling by gap* from $\{1, 2\}$. So suppose that $E(G') \neq \emptyset$ and let f be a *vertex-labeling by gap* from $\{1, 2\}$ for G . Since $G'[X', Y']$ is connected and bipartite, so the color which is induced by f for the set of vertices X' is one and the color which is induced by f for the set of vertices Y' is zero, or vice versa (**Property A**).

For every vertex $v \in X$, consider a variable v in Φ and for every vertex $u \in Y'$ put a clause $(\bigvee_{vu \in E(G)} v)$ in Φ . Now determine whether Φ has a NAE truth assignment. If Φ has a NAE truth assignment Γ , for every vertex $v, v \in X$ label v by 1 if and only if $\Gamma(v) = \text{false}$. Label other vertices of G by 2. It is easy to see that this labeling is a *vertex-labeling by gap* from $\{1, 2\}$ for G . If Φ does not have a NAE truth assignment, then, for every vertex $v \in Y$, consider a variable v in Ψ and for every vertex $u \in X'$ put a clause $(\bigvee_{vu \in E(G)} v)$ in Ψ . Now determine whether Ψ has a NAE truth assignment. If Ψ has a NAE truth assignment Γ by a similar method we can find *vertex-labeling by gap* from $\{1, 2\}$ for G . Otherwise, by Property A, G does not have any *vertex-labeling by gap* from \mathbb{N}_2 . See Algorithm 1.

(ii) We reduce *NAE 3-Sat* to our problem in polynomial time. For a given Φ , we transform Φ into a graph G_Φ such that G_Φ has a *vertex-labeling by gap* from \mathbb{N}_2 if and

Algorithm 1 (Vertex Labeling By Gap From \mathbb{N}_2)

```
1: Input: A planar bipartite graph  $G[X, Y]$ 
2: Question: Does  $G$  have a vertex-labeling by gap from  $\mathbb{N}_2$ ?
3: Let  $G_1, \dots, G_d$  be the connected components of  $G$ .
4:  $j \leftarrow 0$ 
5: for  $i = 1$  to  $i = d$  do
6:    $G'_1[X', Y'] \leftarrow G_1[X, Y]$ 
7:   Remove all vertices of degree one from  $G'_1$ 
8:    $\Phi \leftarrow \emptyset$ 
9:   for every vertex  $u \in V(Y')$  do
10:     $\Phi \leftarrow \Phi \cup (\bigvee_{vu \in E(G)} v)$ 
11:   end for
12:    $\Psi \leftarrow \emptyset$ 
13:   for every vertex  $v \in V(X')$  do
14:     $\Psi \leftarrow \Psi \cup (\bigvee_{vu \in E(G)} u)$ 
15:   end for
16:   if  $\Phi$  has a Not-All-Equal truth assignment then
17:      $j \leftarrow j + 1$ 
18:   else if  $\Psi$  has a Not-All-Equal truth assignment then
19:      $j \leftarrow j + 1$ 
20:   end if
21: end for
22: if  $j = d$  then
23:   return  $G$  has a vertex-labeling by gap from  $\mathbb{N}_2$ .
24: end if
25: return  $G$  does not have any vertex-labeling by gap from  $\mathbb{N}_2$ .
```

only if Φ has a NAE satisfying assignment. Construction of G_Φ is similar to the construction of G_Φ in the proof of (iii) of Theorem 5, except the gadget $P_4 = cc'c''c'''$. For each clause $c \in C$ instead of $P_4 = cc'c''c'''$, put an isolated vertex c . First, suppose that G_Φ has a *vertex-labeling f by gap* from \mathbb{N}_2 and l is the induced proper coloring by f . By an argument similar to argument of proof of Theorem 5, for every clause $c = x \vee y \vee w$, $l(c) = 1$. So $\{f(x), f(y), f(w)\} = \{1, 2\}$, therefore Γ is a NAE satisfying assignment. Now, let Γ be an satisfying assignment for Φ . For every variable x , label the vertex x by 2 if and only if $\Gamma(x) = \text{true}$. This completes the proof.

(iii) The proof is similar to the proof of part (iii) of Theorem 5.

(iv) It was shown that 3-colorability of planar 4-regular graphs is NP-complete [11]. By a proof similar to the proof of part (iii) of Theorem 5, we can prove the theorem. ♣

8 Proof of Theorem 7

(i) We reduce our problem to 2-Sat problem in polynomial time.

2-Sat.

INSTANCE: A 2-Sat formula Φ .

QUESTION: Is there a truth assignment for Φ that satisfies all the clauses?

For a given graph G of order n we construct a 2-Sat formula Φ with n variables v_1, \dots, v_n such that G has a vertex-labeling by degree from $\{1, 2\}$ if and only if there is a truth assignment for Φ . For every edge $e = v_i v_j$, if $d(v_i) = d(v_j)$, add the clauses $v_i \vee v_j$ and $\neg v_i \vee \neg v_j$ and if $d(v_i) = 2d(v_j)$, add the clause $v_i \vee \neg v_j$, otherwise if $2d(v_i) = d(v_j)$, add the clause $\neg v_i \vee v_j$. First, suppose that Γ is a satisfying assignment for Φ . For every vertex v_i , label v_i by 2 if and only if $\Gamma(v_i) = \text{true}$. It is easy to see that this labeling is a vertex-labeling by degree from $\{1, 2\}$. Next, let f be a *vertex-labeling by degree* from $\{1, 2\}$, for every variable v_i , put $\Gamma(v_i) = \text{true}$ if and only if $f(v_i) = 2$. As we know, 2-Sat problem is in P [15]. This completes the proof.

(ii) It was shown that 3-colorability of planar 4-regular graphs is NP-complete [11]. For a given 4-regular graph G and a number k , $k \geq 3$, we construct a graph \mathcal{H}^* with chromatic number $k + 1$ in linear time such that \mathcal{H}^* has a *vertex-labeling by degree* from \mathbb{N}_k if and only if G is 3-colorable. Let $\mathcal{P}_k = \{4, 5, \dots, k\}$.

Construction of \mathcal{H}^* . First, consider a copy of graph G and call its vertices old vertices. For every old vertex v and for every $\alpha \in \mathcal{P}_k$, consider two copies of complete

graphs K_{k-1} . Call this two copies by $K_{k-1}^{(v,\alpha)}$ and $K_{k-1}^{(v,\alpha)'}$. Join v to all vertices of $K_{k-1}^{(v,\alpha)}$ and $K_{k-1}^{(v,\alpha)'}$. Next join all vertices of $K_{k-1}^{(v,\alpha)}$ to a new vertex v_α and also join all vertices of $K_{k-1}^{(v,\alpha)'}$ to a new vertex v'_α . Now, join v_α to v'_α . Call the resulting graph \mathcal{H} . For a given graph \mathcal{H} , for every old vertex v and for every $\alpha \in \mathcal{P}_k$, join every vertex of u , $u \in V(K_{k-1}^{(v,\alpha)}) \cup V(K_{k-1}^{(v,\alpha)'}) \cup \{v_\alpha, v'_\alpha\}$ to $\alpha \times d_{\mathcal{H}}(v) - d_{\mathcal{H}}(u)$ new isolated vertices $l_u^1, l_u^2, \dots, l_u^{\alpha \times d_{\mathcal{H}}(v) - d_{\mathcal{H}}(u)}$. Call the resulting \mathcal{H}^* .

The chromatic number of \mathcal{H}^* is $k+1$. Because every two vertices in $V(K_{k-1}^{(v,\alpha)}) \cup \{v_\alpha\}$ have the same degree, every *vertex-labeling by degree* needs at least k distinct labels. Let f be a *vertex-labeling by degree* from $\{1, \dots, k\}$ for \mathcal{H}^* . We claim that for every old vertex v , $f(v) \in \{1, 2, 3\}$. To the contrary suppose that $f(v) \geq 4$, so there is a $\alpha \in \mathcal{P}_k$ such that $\alpha = f(v)$. Therefore, there is no vertex in $K_{k-1}^{(v,\alpha)}$ with the label 1, hence $l(v_\alpha) = 1$. Similarly, $l(v'_\alpha) = 1$. But v_α and v'_α are adjacent and have the same degree. This is a contradiction. So, for every old vertex v , we have $f(v) \in \{1, 2, 3\}$. Since the degree of old vertices are equal, therefore f is a proper vertex coloring for old vertices, so G is 3-colorable. On the other hand, let c be proper vertex coloring for G with 3 colors. It is easy to find a *vertex-labeling by degree* from $\{1, \dots, k\}$ for \mathcal{H}^* . ♣

9 Proof of Theorem 8

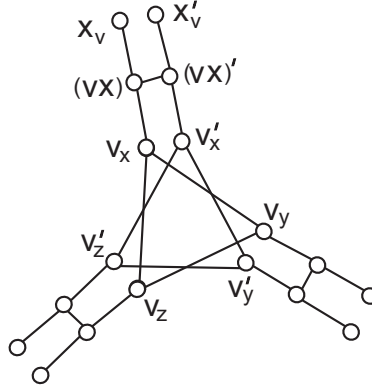


Figure 4: Transformation which is used in constructing G' .

(i) Clearly, the problem is in **NP**. It was shown that it is **NP**-hard to determine the edge chromatic number of a cubic graph [17]. Let G be a 3-regular graph. We construct a 3-regular graph G' from G such that G' has a *vertex-labeling by maximum* from \mathbb{N}_3 if and only if G belongs to *Class 1*. In order to construct G' , for every vertex $v \in V(G)$ with

the neighbors x, y and z consider two disjoint triangles $v_x v_y v_z$ and $v'_x v'_y v'_z$ in G' . Also, for every edge $e \in E(G)$, consider two adjacent vertices e and e' in G' . Finally, for every edge $e = uv \in E(G)$, join e to v_u and u_v ; also join e' to v'_u and u'_v . Name the constructed graph G' (see Figure 4). Since G' has triangles, so every *vertex-labeling by maximum* needs at least 3 distinct labels. First suppose that G' has a *vertex-labeling by maximum* from \mathbb{N}_3 and let ℓ be the induced vertex coloring by f . For every vertex $v \in V(G)$ with the neighbors x, y and z in G , we have $\{\ell(v_x), \ell(v_y), \ell(v_z)\} = \{1, 2, 3\} = \{\ell(v'_x), \ell(v'_y), \ell(v'_z)\}$. Suppose that there are u and v such that $\ell(v_u) = \ell(v'_u) = 3$, since f can not assign 3 to a vertex in a triangle, so $f(vu) = f((vu)') = 3$. Hence $\ell(vu) = \ell((vu)') = 3$ and this is a contradiction. So we have the following fact:

There are no u and v such that $\ell(v_u) = \ell(v'_u) = 3$ (**Fact 1**).

Now, consider the following proper 3-edge coloring for G .

$$g : E(G) \longrightarrow \{1, 2, 3\},$$

$$g(uv) = \begin{cases} 1 & \text{if } f(uv) = 3, \\ 2 & \text{if } f((uv)') = 3, \\ 3 & \text{otherwise.} \end{cases}$$

By Fact 1, g is well-defined and G belongs to *Class 1*. On the other hand, assume that $g : E(G) \longrightarrow \{1, 2, 3\}$ is a proper 3-edge coloring. Define $f : V(G') \longrightarrow \{1, 2, 3\}$ such that for every edge $uv \in E(G)$, $f(v_u) = f(v'_u) = 1$, $f(uv) = g(uv)$ and $f((uv)') \equiv g(uv) + 1 \pmod{3}$. It is easy to see that f is a *vertex-labeling by maximum*.

(ii) First, consider the following simple procedure:

◇ For a given graph G , put a new vertex v and join it to the all vertices of G , next put a new vertex u and join it to v . Name the constructed graph G' .

We can construct G' in polynomial time and G has a *vertex-labeling by maximum* from \mathbb{N}_k if and only if G' has a *vertex-labeling by maximum* from \mathbb{N}_{k+1} . In Part (i), we proved that, it is **NP**-complete to decide whether a given 3-colorable graph G has a *vertex-labeling by maximum* from \mathbb{N}_3 . So by repeating the above procedure, for every k , $k \geq 3$, we can prove that, it is **NP**-complete to decide whether a given k -colorable graph H has a *vertex-labeling by maximum* from \mathbb{N}_k . This completes the proof.

(iii) Consider Algorithm 2. When Algorithm 2 terminates, if it returns " G has the TSV S ", then S is a TSV, so G does not have any *vertex-labeling by maximum*. Suppose that Algorithm 2 returns " G does not have a TSV", but G has a TSV S' . In the lines 2 and 3 of the algorithm, the set of vertices S' are added to S . Now, consider the line 5 of the algorithm and let $v \in S'$ be the first vertex from the set S' that is eliminated from S .

Algorithm 2 (TSV Finder)

```
1: Input: A graph  $G$ 
2: Question: Does  $G$  have a vertex-labeling by maximum?
3:  $S \leftarrow \emptyset$ 
4: for every vertex  $u$  appears in a triangle do
5:    $S \leftarrow S \cup \{u\}$ 
6: end for
7: while there are two adjacent vertices  $u$  and  $v$  such that  $v \in S$ ,  $u \in N(S) \setminus S$  and
   there is no vertex  $z \in S$  such that  $z$  is adjacent to  $v$  and  $u$  or  $v$  does not appear in
   any triangle in  $G[S]$  do
8:    $S \leftarrow S \setminus \{v\}$ 
9: end while
10: if  $S \neq \emptyset$  then
11:   return  $G$  has the TSV  $S$ .
12: else
13:   return  $G$  does not have a TSV.
14: end if
```

When Algorithm 2 chooses the vertex v , v is in a triangle in $G[S']$, so it is in a triangle in $G[S]$. Therefore, there is a vertex u such that $uv \in E(G)$, $v \in S'$, $u \in N(S) \setminus S$ and there is no vertex $z \in S$ such that z is adjacent to v and u . So S' is not a TSV. This is a contradiction. So when Algorithm 2 returns " G has no TSV", G does not have any TSV.



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